Nodal analysis of finite square resistive grids and the teaching effectiveness of students' projects

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ABSTRACT: In this paper, the issue of teaching the node voltage method by a student's project is presented. In the first part, basic information about the subject of the project is presented and discussed. This concerns the computing of the resultant resistance between two chosen nodes of a resistive grid. In the next part, the authors describe assumptions of the project given to undergraduate students. Then, students' results and improvements to the project, as well as students' involvement are discussed. In conclusion, the authors elaborate on how the students' projects improved the effectiveness of the teaching of the method.

INTRODUCTION

In the years 2008–2010, students at the AGH University of Science and Technology, Department of Electrical Engineering, specialising in Computer Engineering in Electrical Systems, during the 7th semester of their studies, participated in lectures on the subject, Computer-aided Analysis of Electronic Systems.

Both lectures and laboratory/project activities of that subject were carried out in the English language. Semester credit was built using the marks of five projects, which were prepared during laboratory activities.

The object of the first project was the nodal analysis of a finite square resistive grid on a plane. Students implemented an algorithm for computing the resistance between two nodes of a grid, based on the node voltage method. In the next step of the project, students had to compare different methods for solving the resulting set of equations.

The authors could compare the effectiveness of teaching the node voltage method, with the same method as taught to students of this Department earlier, during the 2nd and 3rd semesters of their studies in the subject, Electrical Circuit Theory. In this case, students participate only in lectures and classes, where they solve some problems theoretically, and only learn how to write down the set of equations for a given circuit.

FINITE SQUARE RESISTIVE GRIDS ON A PLANE

Consider finite square resistive grids on a plane. Resistive grids are electrical circuits in which the resistive elements connect neighbouring points of a square lattice. The case will be considered where the nodes fill a square area of the plane. It is assumed that each interior node is connected with all of its four neighbours by means of a conductance, which is finite and positive, while each boundary node is connected only to one interior node (Figure 1).

The assumption that boundary nodes are not connected to each other is not a limitation. If a grid contains connections at the edges, auxiliary nodes can be added around the network to connect them with the network by means of a fixed conductance to obtain a grid with the shape shown in Figure 1.

The size of a grid is defined as the number of rows (or columns). The square grid shown in Figure 1 is of size 7. It can be seen that this grid has 25 interior nodes and 4 groups of 5 boundary nodes. In general, a grid with size n has $(n-2)^2$ interior nodes and $4(n-2)$ boundary nodes, which gives a total number of (n^2-4) nodes [2].

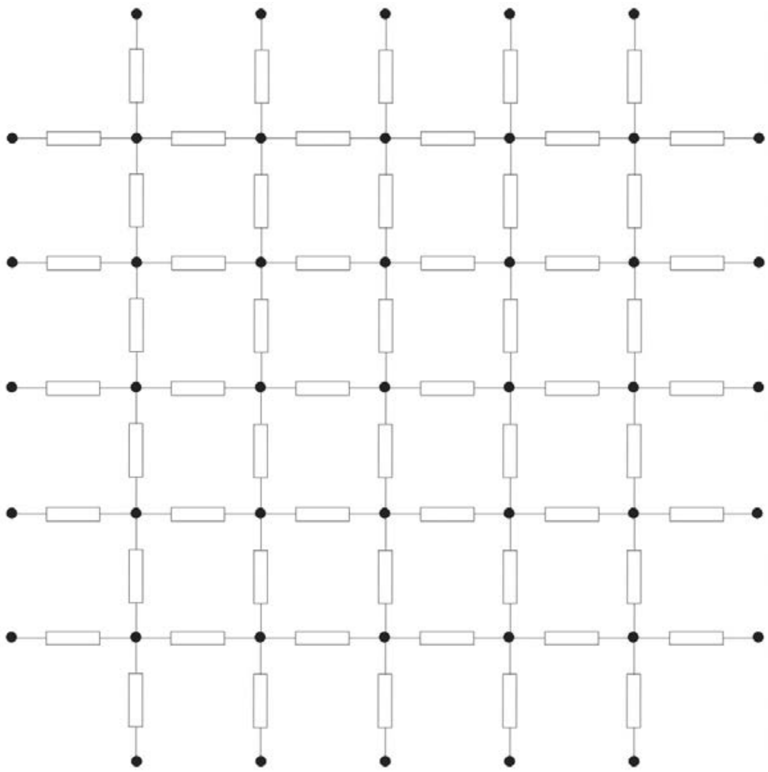

Figure 1: Example of finite square resistive grid [2].

CALCULATION OF RESISTANCE - FORMULATION OF THE PROBLEM

Consider the set of $m \cdot n$ nodes arranged in n columns and m rows (see Fig. 2.) The potential of the node lying in the i-th row and the j-th column is V_{j}^{i}. If (i,j) and (k,l) are neighbouring nodes, the equation $|i-k|+|j-l|=1$ is true. Call the resistance connecting nodes (i,j) and (k,l) R_{kl}^{ij}. The current injected into the node (i,j) is $I_{i,j}$. Assume, that the sum of all currents for all nodes of the grid is equal to zero, which follows the Kirchhoff's Current Law. Consider the problem of calculating the potential at the nodes V_{j}^{i} for given values of resistance R_{kl}^{ij} and currents $I_{i,j}$.

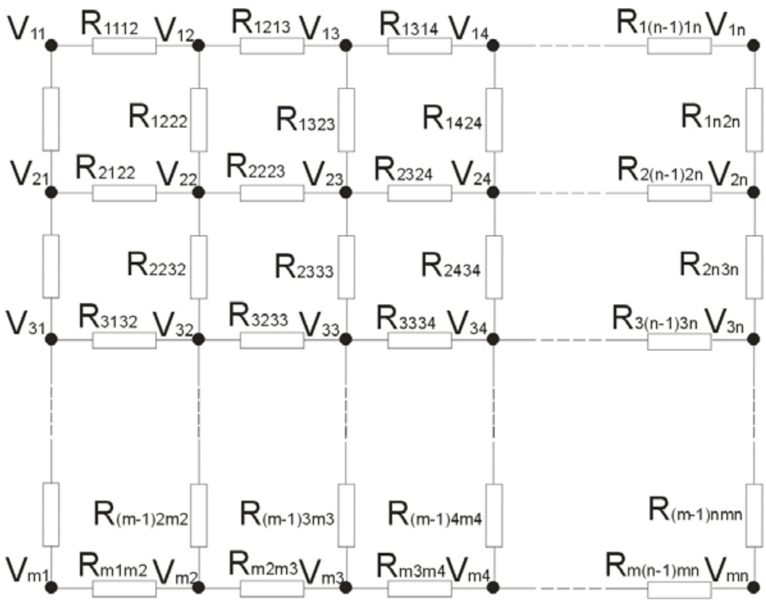

Figure 2: Finite square resistive grid of size m x n [1].

Calculating the resistance between two chosen points of the grid is a special case of the problem above. Choosing the nodal analysis method to find the potentials at the nodes of the grid, Kirchhoff's Current Law for a node (*i*,*j*) is:

$$G_{i,j,i+1,j}(V_{i,j}-V_{i+1,j})+G_{i,j,i-1,j}(V_{i,j}-V_{i-1,j})+G_{i,j,i,j+1}(V_{i,j}-V_{i,j+1})+G_{i,j,i,j-1}(V_{i,j}-V_{i,j-1})=I_{i,j} \tag{1}$$

This set of equations does not have a unique solution – changing the potential of one node by a fixed value does not change the currents in all branches of the circuit. Fixing the potential of a chosen node, i.e. $V_n^m=0$, and deleting the appropriate equation gives a linear set of equations with a unique solution. In matrix form, the set of equations can be written as:

$$G\cdot V=I \tag{2}$$

where G is the conductance matrix resulting from the topology of the grid, $V=(V_{1,1},V_{1,2},V_{1,3},\dots,V_{1,n},V_{2,1}\dots,V_{m,n})$ is a vector of potentials at the nodes of the grid, and I is a vector of the currents injected at the nodes.

The problem of the calculation of the resultant resistance seen from the nodes *(i,j)* and *(k,l)* is reduced to solving the set of equations above, for the case where two elements of the vector *I* are non-zero: $I_{i,j}=-I_{k,l}=i$. Resultant resistance is a quotient of a difference of the potentials $V_j^i - V_l^k$ and the value of the current *i*.

CONDUCTANCE MATRIX – CHARACTERISTIC PROPERTIES

When constructing algorithms for solving a given problem, the specific properties of the conductance matrix for the resistive grid are very important. It is a square matrix with size $n\cdot m-1$, which means that even for small girds we have a large matrix. Furthermore, this is a band matrix, with the bandwidth $2n+1$. In each row and each column there are no more than five non zero elements, i.e. when all resistances in a grid of size $n=m=4$ has a value of 1, the set of equations is:

$$\begin{bmatrix}
2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -1 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 0 & 0 & 0 & 3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 3 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 3 & -1 & 0 & 0 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 3 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 3
\end{bmatrix}
\cdot
\begin{bmatrix} V_{11}\\ V_{12}\\ V_{13}\\ V_{14}\\ V_{21}\\ V_{22}\\ V_{23}\\ V_{24}\\ V_{31}\\ V_{32}\\ V_{33}\\ V_{34}\\ V_{41}\\ V_{42}\\ V_{43} \end{bmatrix}
=
\begin{bmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 4\\ 0\\ 0\\ -4\\ 0\\ 0\\ 0\\ 0\\ 0 \end{bmatrix} \tag{3}$$

The observations above mean that for solving this specific problem, sparse matrix methods can be used. Furthermore, the elements on the main diagonal are larger than, or equal to, the sum of the magnitudes of the rest of the elements in the same row (or column). Therefore, the G matrix is diagonally dominant for the rows and the columns of the matrix.

STUDENTS' PROJECTS – REALISATION

As mentioned in the introduction, students in the 7th semester of Electrical Engineering at the AGH University of Science and Technology participated in lectures on the subject, Computer-aided Analysis of Electronic Systems. The lectures were led by Prof. Zbigniew Galias, and the laboratory/project activities were led by one of the authors, Piotr Zegarmistrz, both employed in the Department of Electrical and Power Engineering. In the years 2008-2010, there were more than 50 participants in the lectures.

The first task of the laboratory activities was to implement an algorithm for computing the resistance between two chosen nodes of the grid, using nodal analysis. Firstly, the teacher presented a theoretical introduction to the project. Information given to the students was similar to that above in this paper. Then, students implemented their own algorithm by writing down the set of equations (in particular, the conductance matrix *G*), describing the given problem.

Choosing the software solution for that was an individual decision of the student. Most of them have chosen MATLAB software, but some decided to implement their own application, using a well known programming language (mostly C++ or Java).

The simplest version of the algorithm assumed the user only chooses the size of the grid with all resistances having a fixed value of 1. The implementation of the algorithm led to the creation of an application allowing the user to change resistances, which then creates the conductance matrix *G* for a given grid (with both grid size and element values set by the user).

In the next step the user chooses nodes, between which the resultant resistance will be calculated. In practice, it is realised by connecting the chosen nodes with current sources of a fixed value. In the application, the problem is restricted by having the current vector *I* with only two non-zero elements (the same value, different sign) in positions corresponding to the chosen nodes. When the conductance matrix *G* and current vector *I* are created, there is enough information to solve the set of equations and find the vector *V* of the potentials at all of the nodes.

The method of solving a linear set of equations was freely chosen by the students. Some, who had decided to implement their own application, were forced to create functions, which could then be used to find the solution. Most had chosen methods such as Gauss-Jordan elimination or LU factorisation. The rest, who had chosen the MATLAB software, tried to compare the effectiveness of the MATLAB functions with their own algorithms. The measure of the effectiveness was the time required to compute the solution. Furthermore, all of the students tried to show the relationship between the size of the grid and the time to compute. Some also tried to prove that there exists a limit of resistance between two chosen nodes, for an infinite size grid with elements with resistance value of 1. This was undertaken by the calculation of the resultant resistance for two nodes with the grid growing in size.

STUDENTS' PROJECTS – RESULTS

The most interesting results found in the reports from students' projects are presented below. The authors have chosen one example for each object of research mentioned in the previous part of this paper. It should be emphasised that almost 50 reports from students' projects were analysed.

Student Janusz Duc (Electrical Engineering 2006-2011) showed the relation between the number of nodes in the network and the operation time for the MATLAB function (see Reference [5]). Results of the students' research are shown in Figure 3 below.

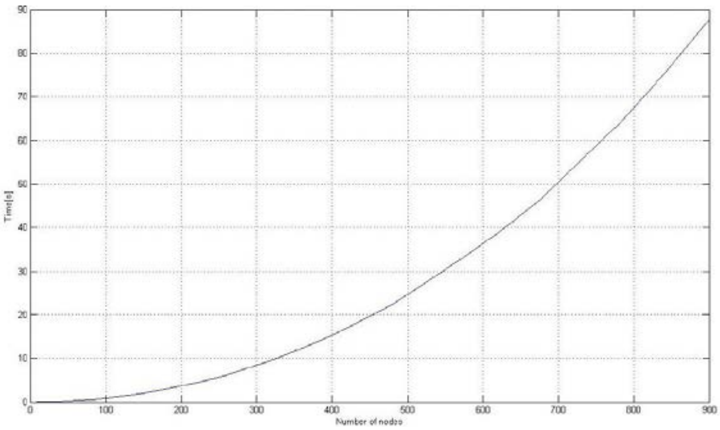

Figure 3: Dependence between the number of nodes in the network and the operation time - Janusz Duc, 29.10.2009 [5].

In the same paper, there is included a very interesting comparison of the MATLAB function for solving the set of equations with the student's own algorithm based on the Gauss-Jordan elimination method. Table 1 presents the results of that research.

Table 1: Comparison of computing time for the chosen methods - Janusz Duc, 29.10.2009 [5].

size of network	Matlab function rref		Gauss–Jordan elimination – own algorithm	
	time [s]	resistance [Ω]	time [s]	resistance [Ω]
10 x 10	0.9048	2.7093	0.6552	2.7094
20 x 20	14.4457	3.5905	347.0086	3.5899
30 x 30	83.2577	4.1091		
40 x 40	308.1488	4.4729		
50 x 50	909.9538	4.7556		
60 x 60	2346.4	5.0000		

Student Dariusz Kowalik (Electrical Engineering 2006-2011) presented the dependence between the resultant resistance for two chosen nodes with fixed placement and grid size, and the grid, where all resistances has a value equal to 1. He also showed how the resistance depends on the position in the grid, and that this dependence decreases with the grid size. This proves that there exists a limit of resistance for infinite grids. Figure 4 presents that part of the student's report.

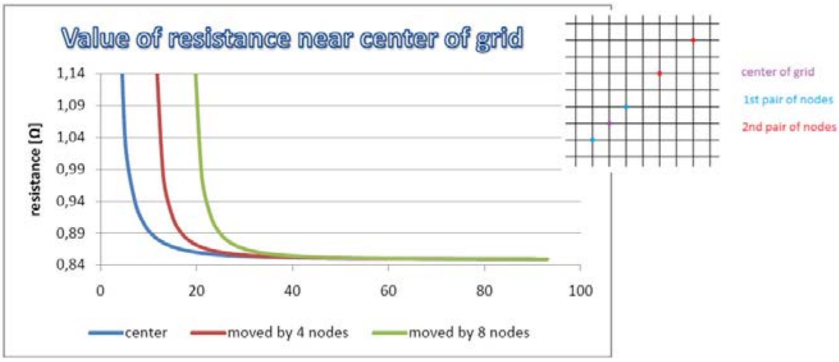

Figure 4: Limit of the resistance for infinite grid - Dariusz Kowalik, 08.11.2009 [6].

As a comment on Figure 4, the student wrote: *The chart represents values of resistance measured at 3 points: in the centre of the grid, away from the centre in the diagonal direction by 4 nodes, and by 8 nodes. The value of the resistance is increased away from the grid's centre. In small grids, the resistance strongly depends on the distance from the centre. The greater the grid's size, the smaller the resistance dependence on distance* [6].

The same student presented a very interesting comparison of the mathematical methods implemented by himself. As mentioned earlier, as the measure of effectiveness was the time required to compute the solution. It is clear, that the time required to compute depended on the machine used. Therefore, the exact time of computation itself does not matter, but the relation between results for different methods on a given machine is important.

The student tested Gauss-Jordan elimination, LU factorisation with partial pivoting, matrix left division, using an inverted matrix and a sparse matrix. A sparse matrix is a particular way of keeping a matrix in memory. Unlike the common way, where all elements of a matrix are kept in memory, with a sparse matrix only coordinates and values of non-zero elements of matrix are kept in memory, which is more memory efficient [6].

Table 2: Results of simulation for 30x30 resistive grid, comparison of methods - Dariusz Kowalik, 08.11.2009 [6].

	Gauss-Jordan elimination	LU factorization	matrix left division	using inverted matrix	sparse matrix left division	using sparse inverted matrix
time [s]	71.191322	0.224325	0.127576	0.764821	0.016648	0.099578
resistance [Ω]	4.1091	4.1058	4.1058	4.1058	4.1058	4.1058

In conclusion, the student noted: *As it can be seen above, the Gauss-Jordan elimination is a hundred times slower and inaccurate by comparison with other methods. Therefore, taking into consideration the speed of computation and memory efficiency, the sparse matrix left division will be used* [6].

Student Grzegorz Gancarczyk (Electrical Engineering 2005-2010) decided to check the correctness of calculations by comparing his results from MATLAB with a simulation of the circuit using PSPICE software. He built a small grid (3x3) in the PSPICE environment, and compared the resulting potentials of the nodes. Results of this experiment are shown on Table 3.

Table 3: Comparison of results, MATLAB and PSPICE - Grzegorz Gancarczyk, 17.11.2008 [4].

	OUTPUT: 1-9		OUTPUT: 3-5	
Node No.	MATLAB	PSPICE	MATLAB	PSPICE
1	0,75	.7500	-2,2879e-16	-69.39E-18
2	0,875	.8750	0,125	.1250
3	1,25	1.2500	0,625	.6250
4	0,625	.6250	-0,125	-.1250
5	0,625	.6250	-0,25	-.2500
6	0,625	.6250	0,125	.1250
7	0,5	.5000	-0,125	-.1250
8	0,375	.3750	-0,125	-.1250

CONCLUSIONS

Nodal analysis of electrical circuits is not very easy for students to learn. Academic teachers' experience shows that students choose that method only when forced to, when solving problems in Electrical Circuit Theory. Probably, they try to avoid methods, which they do not understand.

A much more popular method is to use the superposition theorem, the mesh current method or simply to write down the set of equations resulting from Ohm's and Kirchhoff's Laws. Students say those methods are more intuitive. In addition, it is much easier to check the correctness of the written equations. It is not so easy to realise that, when checking the correctness of node voltage equations, it is necessary to sum the currents in the node, and then multiply the potential differences and conductance.

The observations of the authors are that a student's project with an interesting problem to solve is more effective in teaching the method than a theoretical solution consisting of many little tasks for simple electrical circuits solved on paper or a board. When students worked on the implementation of the algorithm, and wrote down the conductance matrix *G* for a given problem, they easily understood how the method worked, and how it s implified deriving the solution.

Another conclusion is that this type of project can change a student's decision when choosing the method to solve the problem. Up until then, they tried to solve all circuit theory tasks beginning from the mesh current method. Now, they check to see, if the node voltage method gives an easier solution.

It is also very important that, if the project problem is interestingly presented, the teacher observes high student involvement; that students propose their own improvements to the algorithm, and try to find some interesting results and conclusions from their computations.

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