Volume 15, Number 2, 2013

Global Journal of Engineering Education

Engineering students' visual thinking of the concept of definite integral

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ABSTRACT: The purpose of this study is not only to extend our understanding of students' difficulties and strengths associated with visualisation, but also to identify the types of visual image they utilise while solving integral problems. Through detailed analyses of students' work and verbal protocols, high proportions of students with high visualisation ability use imagination images along with algebraic representations, and linking these two representations leads them to successful problem solving. Students with low visualisation ability use memory images. The study shows that students can produce imagination images that play a significant role in the problem-solving process. As such, a process of visualisation allows an articulation between representations to produce another representation that could help students to solve given problems.

Keywords: Engineering students, definite integral, representation, visual thinking, visualisation

INTRODUCTION

The research literature on mathematics education has long discussed the merits of visualisation and analysis in mathematical thinking. Visualisation has been an area of interest for a number of researchers concerned with mathematics education. Many researchers emphasise the importance of visualisation and visual reasoning for learning mathematics, and visualisation is a fundamental aspect in understanding students' construction of mathematical concepts [1-3]. In their arguments for visualisation, authors suggest that visual thinking can be an alternative and powerful resource for students doing mathematics, a resource that can open the door to ways of thinking about mathematics other than the linguistic and logico-propositional thinking of traditional proofs and the symbol manipulation of traditional algebra. According to Zimmermann, *...the role of visual thinking is so fundamental to the understanding of calculus that is difficult to imagine a successful calculus course which does not emphasise the visual elements of the subject* [4]. Although visualisation must necessarily play an important role in mathematical activity, it clearly needs research which helps understand more about which of its features contribute significantly to the role in a given mathematical situation. Many studies have focused on derivative and antiderivative graphs, but there has been little research on the concept of definite integral [5][6]. This study stands apart from other research on learning calculus, because it not only extends the understanding of students' difficulties and strengths associated with visualisation, but also identifies the types of visual image they utilised while solve integral problems.

THEORETICAL FRAMEWORK

Representation is an indispensable tool for presenting mathematical concepts, communicating and considering or thinking. Hiebert and Carpenter assessed students' understanding of concepts based on the relationships between the representations they created [7]. They contended that *...the mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and the strength of the connections* [7]. This perspective supports the one proposed by Duval [8]. Duval maintained that the process of mathematical thinking required not only the use of representation systems (which Duval called registers) but also cognitive integration of representation systems. Based on Duval's analysis, learning and comprehending mathematics require relatively similar semiotic representations. From this perspective, the understanding of a mathematical concept is built through tasks that imply the use of different systems of representation and promote the flexible coordination between representations. Therefore, learning mathematics implies *...the construction of a cognitive structure by which* *the students can recognise the same object through different representations* [8]. The learning of calculus movement emphasises the use of multiple representations in the presentation of concepts, that concepts should be represented numerically, algebraically, graphically and verbally wherever possible, so that students understand connections between different representations and develop deeper and more robust understanding of the concepts.

The essence of the concept of integrals (including other mathematical concepts) is that the process concept and object concept can be presented by connected but different representations. A number of studies have indicated that the representations used by students to solve an integral problem are related to the meanings they attribute to the concept of integrals [9]. The graphical representation of definite integrals is typically used in calculations that involve areas under a curve, whereas numerical representations are used for Riemann's cumulative addition problems [10]. Solving integrals using common integration techniques demonstrates the need for symbolic representations.

In this study, the author situated his investigation of representation theory within the context of integrals. Specifically, the author has examined students' ability to use the relationship between representations to solve integral problems. In this specific case - the understanding of the concept of integral - research conducted with this representational approach highlights as a cause of these difficulties the lack of coordination between both the graphic and algebraic representations and the predominance of the latter in the students' answers. This leads the author to pay special attention to the use of the graphical representation and to visualisation.

Visualisation is a critical aspect of mathematical thinking, understanding and reasoning. Researchers argue that visual thinking is an alternative and powerful resource for students to do mathematics; it is different from linguistic, logic-propositional thinking and the manipulation of symbols. A growing body of research supports the assertion that understanding mathematics is strongly related to the ability to use visual and analytic thinking. Researchers contend that in order for students to construct a rich understanding of mathematical concepts, both visual and analytic reasoning must be present and integrated [11][12].

Visualisation involves both external and internal representations (or images) and, thus, following Presmeg [13], one can define visualisation as processes involved in constructing and transforming both visual images and all of the representations of a spatial nature that may be used in drawing figures or constructing or manipulating them with pencil and paper. This definition emphasises that in mathematical thinking and problem solving, an appropriate graph can be drawn to represent the mathematical concept or question, and that the graph can be used to understand a concept or as a problem-solving tool. In this study, the author investigated the visual images that students used to solve specific problems and how they managed the given visualisations.

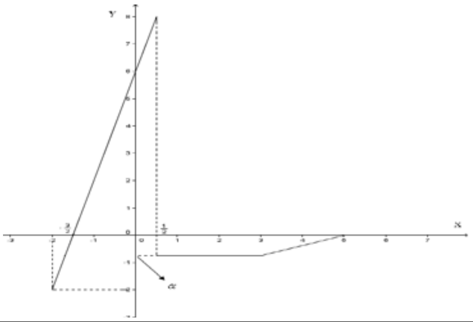
METHODOLOGY

Participants and Instruments

The 15 first-year engineering students who participated in this study were enrolled at a university of technology, and had learned the basic rules of integration using primitives, as well as their relationship to the calculation of a number of areas under curves. These students' calculus grades were in the top 10% among 352 students. The instruments used for data collection were a questionnaire containing problems and interviews. The questionnaire comprised five problems in definite integral (Figure 1), some of which were referenced from other studies [14][15].

Task 1. If $\int_1^3 f(t)dt = 8.6$, use two strategies to evaluate the value of $\int_2^4 f(t-1)dt$.

Task 2. The graph of f is sketched below. Given that $\int_{-2}^{5} f(x)dx = \frac{39}{8}$, determine the value of α.



Task 3. Is it true or false that if $\int_a^b f(x)dx \geq \int_a^b g(x)dx$, the $f(x) \geq g(x)$ for all $x \in [a, b]$? Justify your answer.

Task 4. If $\int_1^5 f(x)dx = 10$, use two strategies to evaluate the value of $\int_1^5 (f(x)+2)dx$.

Task 5. Use two strategies to calculate $\int_{-3}^{3} |x+2| dx$

Figure 1: The study questionnaire.

These problems enabled the students' performance regarding the visual thinking to be analysed. The results of the questionnaire necessitated further investigation into the visual thinking of the students. The clinical interviews were carried out after the answers to the problems had been analysed [16].

Each interview lasted about 40-50 minutes and was video- and audio-taped. In order to prepare the script for the interview, the author analysed the written answers focused on to how the students seemed to use and coordinate the different mathematical representations needed. During each interview the students were asked to think aloud, while they were solving the tasks so that the author could describe their responses and strategies, as well as make inferences about their mental processes and images.

The analysis focused on identifying the students' mental processes and images used to create meanings for the problems and the justifications provided. For each problem, the author identified the following:

- The visual images and representations that the student used.
- The relationships that the student established between the visual images and representations to generate new information.

This analysis provides information on how students use and relates the representations of mathematical knowledge to obtain new information.

To assess and interpret the visual thinking of the definite integral concept of students in this study, the author constructed the visual thinking structure of definite integrals, employed this structure to develop clear standards and, then, classified the visual thinking of the students into various levels. The standards were related to the thought process adopted and presented by the students when solving problem, as well as their potential to construct relationships among the various representations and properties, and the degree to which they integrated these relationships into their explanation of problem solving. The analysis results indicated that the visual thinking distribution of 15 students could be categorised into five competencies and three levels. In the non-visual (NV) level, one tends to focus on a single visual image, overlooking other representations of a similar nature.

For example, the students believed that they had to perform integral operations to determine $\int_a^b f(x)dx$. In the local-visual (LV) level, one can perceive and confirm the relationships among various visual images; however, these items may still appear independent of each other. For example, students understood the relationship between the integral and area, but were unable to differentiate between the relationships of areas above and below the x axis with the integral. In the global-visual (GV) level, one can use the relationships to construct a consistent structure based on the relationships among various visual images. The importance of this consistency lies in the fact that it determines the scope of visual thinking.

RESULTS AND DISCUSSION

According to the data analysis, the author identified five competencies of visual thinking relating to the concept of definite integral and, then, classified the visual thinking of the students into three levels (Table 1). Because students' visual thinking of the definite integral could be reasonably understood regarding the three levels, the author evaluated the responses to the task interviews, searching for evidence of the NV, LV and GV levels. Most students appeared to be in either the NV or LV levels, and the difference was usually obvious, as demonstrated by the representative examples included in the following discussion.

Table 1: Visual thinking of three levels.

Visual thinking abilities	NV	LV	GV
To understand algebra and geometry as alternative languages	×	○	○
To extract specific information from diagrams	×	○	○
To represent and interpret problem (or concept) graphically	×	∆	○
To draw and use diagrams as an aid in problem solving	×	∆	○
To understand mathematical transformations visually	×	○	○
To have in mind a repertoire of important visual images	×	×	∆

× denotes that students have not demonstrated the competency in the problem solving

∆ denotes that students have not demonstrated the competency fully in the problem solving

○ denotes that students demonstrated the competency completely in the problem solving

Visual Thinking of the Concept of Definite Integral in the NV Level

The initial level of visual thinking is the non-visual level. The author categorised four students into this group. One of the visual thinking characteristics shared by these students was that they could not recognise the relationship between the area and integral. These students could only process representations within a representation system, and the representations used were influenced by the representation format employed for problems. Additionally, they preferred solving problems using symbolic representations.

The students could solve a number of problems by simply applying rules that had been memorised and, in some cases, incorrectly remembered. Consider the following excerpt from the interview conducted with Porter, who has a collection of rules that enable him to integrate fundamental functions, such as the integrals in Tasks 4 and 5. However, he could not solve the problems using graphical representations.

In Task 3, he stated that the proposition was true and provided specific examples of functions as evidence without giving graphic representations, failing to provide suitable justifications. He provided a specific example that defined two functions $\mathrm{f(x)} = x^2$+1 and $\mathrm{g(x)} = x^2$, and calculated two integrals between x = 1 and x = 2 to obtain 10/3 for f and 7/3 for g. When drawing graphs based on the two functions provided, Porter was unable to think using graphical representations without algebraic formulae. Therefore, Porter's thinking style tended to be analytical but not visual; he did not understand algebra and geometry as alternative languages, did not represent and interpret the problem graphically and did not understand mathematical transformations visually. Porter used visual representations as memory images and the graphical representations in his memory as standard graphics. An image of a standard figure may cause inflexible thinking, which prevents the construction of a non-standard diagram.

The students at the non-visual level generally used a single representation and a symbolic representation was used to solve all types of problems. This indicates that the students consider symbolic representation as a support tool. Additionally, the students in this group were inclined to rely on analytical thinking instead of visual thinking. This leads to a tendency to be cognitively fixed on standard figures and procedures instead of recognising the advantages of visualising the tasks. Presmeg showed that visualisation could be a hindrance for solving a mathematical problem, especially when a mental image of a specific subject controls the student's thinking. In this group of students' cases, the mental image of a standard figure has dominated their thinking when trying to draw a figure to solve problems [13].

Visual Thinking of the Concept of Definite Integral in the LV Level

The next level of visual thinking of the concept of definite integral regarding the existence of cognitive links and awareness of these links is the local-vision level. The author categorised nine students into this group. These students understood the relationships between representation systems and could change or transfer the representations in some of the representation systems. However, these students had difficulty coordinating these relationships. For example, Helen was one of the students in this group. She could use correct symbolic representations to perform mathematical thinking and could manipulate the area using graphical representations according to the changes in integral symbols in Tasks 1, 4 and 5. Consider Task 1 for example, Helen assumed that $\mathrm{F'(t) = f(t)}$, then $\int_1^3 \mathrm{f(t)dt = F(3) - F(1) = 8.6}$. Consequently, $\int_2^4 \mathrm{f(t-1)dt = F(3) - F(1) = 8.6}$. Helen knows that because the two integrals represent the same area, so the two integrals have the same values.

The results show that the students in this group have begun to coordinate the representation systems of definite integrals. These students can perform representation transformations in separate representation systems. The students in this group differ from those at the non-visual level in that they have understood algebra and geometry as alternative languages for the concept of definite integral, and developed visual methods to *see* mathematical concepts and problems better. The students in this group used visual representations as induced image and they induced the visual images mainly from the analytic thinking. Although their visual thinking inclines toward local not global thinking, this restricted visualisation actually hinders their solving of the tasks. Additionally, their chosen visualisation only reflects one aspect of the integral concept, which has a number of significant consequences on their ability to solve the other tasks. According to Zimmermann, an important component of visual thinking is the ability to recognise that an answer obtained algebraically is false based on geometric grounds; the interviews show that this component is not present in the minds of many students [4].

Visual Thinking of the Concept of Definite Integral at the GV Level

Two students were categorised into this group. These students could recognise the relationships among representation systems and convert representations between representation systems. In Tasks 1, 4 and 5, Kevin used the correct symbolic representations to perform mathematical thinking. He also manipulated the area using graphical representations according to the changes in integral symbols. Consider Task 4 for example, Kevin actually employed three methods to solve the problem. The first method was the standard algorithm $\int_1^5 \mathrm{(f(x)+2)dx} = \int_1^5 \mathrm{f(x)dx} + \int_1^5 \mathrm{2dx} = 18$; the second method was the mean value theorem for integrals

$f_{ave} = 1/4 \int_1^5 f(x)dx = \frac{5}{2}$, $\int_1^5 (f(x) + 2)dx = (f_{ave} + 2) \times (5 - 1) = 18$; and the third method was a graphical representation. He knows how to draw the graph of f(x) + 2, and understands the relationship between the integrals and areas. For Task 3, Kevin stated that the proposition was false and provided graphical representations (Figure 2). Subsequently, the interview progressed as shown below.

R: If f(x) is greater than g(x), would the integral of f(x) be greater than the integral of g(x)?
K: Yes.
R: Why?
K: If f(x) is greater than g(x), the difference of f(x) minus g(x) would be greater than 0 and a positive value. The integral of f(x) minus g(x) would be a positive value; therefore, the integral of f(x) would be greater than the integral of g(x).
R: Why do you think Task 3 is incorrect?
K: The situation in Task 3 is opposite to that of your question. The integral values of the function in the interval [a, b] are greater, and the values of the function in the interval [a, b] are not necessarily greater than that of g(x).
R: Why is that?
K: In this graph I drew, the area below f is greater than the area below g in the interval [a, b]; therefore, the integral of f in [a, b] is greater than the integral of g in [a, b]. However, the function value of f in the interval [a, c] is smaller than the function value of g in the interval [a, c]. Therefore, the description in this task is incorrect.

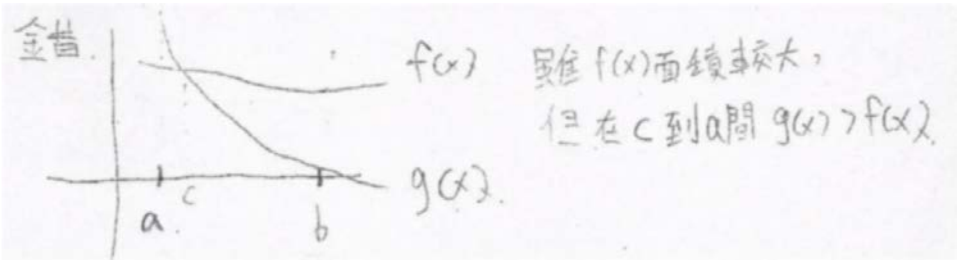


Figure 2: Kevin's graphical presentation for Task 3.

Unlike students at the non-visual level who can only apply symbolic representation thinking, Kevin could employ graphical representation as a thinking tool. Additionally, Kevin clearly understood that the area above the x axis was the integral value, and he understood the relationship between the area under the x axis and integral (Task 2, Figure 3).

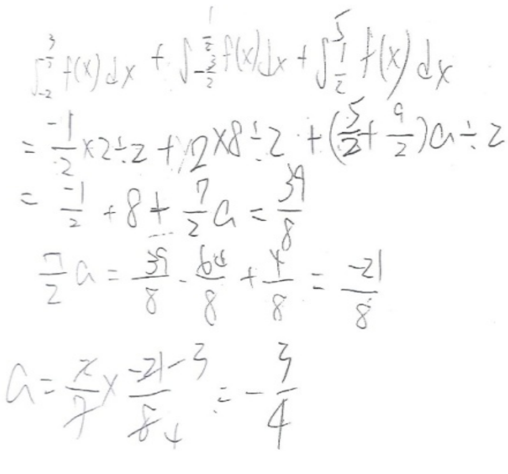


Figure 3: Kevin's problem-solving process for Task 2.

Kevin clearly extracted specific information from diagrams, represented and interpreted the problem (or concept) graphically and understood mathematical transformations visually. The most significant differences between this group and students in the other groups were that they had the ability to perform representation treatments in representation systems, and they could perform representation conversions among various representation systems. On the other hand, the visual images that this group of students used are imagination images, some kind of spontaneous, self-constructed, non-structural, a new type of organisation, to integrate past and current experience in the creation of things.

Imagination images are different from memory images to students, because imagination images did not really exist in the past, but were generated through the students' creative process. The added accuracy in this group of students' drawings could be characterised as *introducing suitable notation*, which was Pólya's recommendation when using visual representations [17]. For Kevin, visualisation is a powerful tool for exploring mathematical problems and for ascribing meaning to the concept of definite integral and the relationship between them. Additionally, visualisation reduces the complexity when considering a significant amount of information.

CONCLUSIONS

In this study, the author recruited first-year university of technology calculus students to be research participants to investigate visual thinking of the definite integral concept. Visual thinking involves the capacity to make connections between both mathematical objects and concepts, and mathematics and the physical world. The data analysis results show that the main obstacles preventing students from freely shifting within the representation system for the concept of definite integrals were that they did not have the ability that involves visualising the abstracted relationships and non-figural information into visual representations and imagery. The development of visualisation ability, which may influence the relationship between graphical representations and the other representations, increases the performance of solving definite integral problems.
For students with non-visualisation ability, the visual images used are memory images, whereas a high percentage of students with low visualisation ability use algebraic representations or use graphical representations that are induced from analytical thinking and this leads to serious difficulties in problem solving. Possible reasons for this tendency include lecturers' reliance on a single representation in their teaching and lecturers' reluctance to show examples that enable the use of multiple representations, leading to students' lack of knowledge of alternative definitions.

A considerable percentage of students with high visualisation ability use graphical representations along with algebraic representations so that linking these representations leads to success in solving the problem. Aspinwall et al quote MacFarlane Smith as saying that *gifted individuals have their own internal* blackboards *and can visualise complicated structures without being aware that they are doing so* [18]. Although not necessarily gifted, the idea of internal blackboard seems to apply to Kevin.

From a didactic perspective, the introduction of graphics that illustrate a specific case and a counter-example may focus attention on the key aspects of the representational relationship and lead to the graphic becoming an integral part of the concept of definite integrals. However, adopting a specific method of teaching visual thinking raises the following questions: What conversion processes are involved in moving among various mathematical representations, including those of a visual nature? How can visual, mixed and non-visual methods be combined in class to improve the visual thinking of students?

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BIOGRAPHY

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