Investigating engineering students' versatile thinking

Chih-Hsien Huang

Ming Chi University of Technology
New Taipei City, Taiwan

ABSTRACT: A study of students' versatile thinking and the learning of the concept of integration in mathematics are reported in this article. By applying the concept of versatile thinking as the research guide to a specific framework for examining mathematics education in universities, students' responses to interview questions were analysed and interpreted. Versatile thinking comprised process/object, visual/analytic and representational versatility, which included: the ability to work within a representation system; transfer seamlessly between the systems of specific concepts; engage in procedural and conceptual interaction with specific representations; the visual images students use to resolve specific problems; and how students handle specific visualisations. Analysis of the interviews and results indicated that co-ordination between the concept of the graphic representations and the visual ability of the integral problems is necessary for excellent versatile thinking regarding the concept of definite integrals.

INTRODUCTION

Integration is a core concept in the calculus curricula and also occurs in languages and tools from other fields, such as physics, engineering, economics and statistics. Mathematics education research communities have extensively discussed students' learning and the development of mathematical concepts. Topics that have been discussed include the concepts of process and object [1][2] and different representations [3]. Students' ability to convert process-objects and representations in mathematical concepts require and involve flexible thinking about mathematical concepts. Regarding engineering education, numerous studies have highlighted that mathematics instruction for engineering students should comprise not only mathematical knowledge but also training in mathematical thinking [4][5]. The results showed that training in mathematical thinking is the most important objective of university mathematics education.

Flexibility of thinking is essential to the process of mathematical learning. Thomas proposed versatile thinking, emphasised the effectiveness of cognitive activities and identified the primary factors that influence problem-solving strategies. Versatile thinking comprises the following three elements: a) representational versatility; b) process-object versatility; and c) visuo-analytic versatility [6]. Although the structure of versatile thinking has been applied recently to mathematical and statistical learning, previous studies primarily focused on one or two of the elements [7][8]. Studies that have investigated the learning and teaching of mathematical concepts by combining the three elements are rare. The work reported in this study, used this three-element structure as the theoretical framework to investigate the versatile thinking of engineering students regarding the concept of integrals.

THEORETICAL FRAMEWORK

Representation is an indispensable tool for presenting mathematical concepts, communicating, considering or thinking. Hiebert and Carpenter assessed students' understanding of concepts based on the relationships between the representations they created [9]. This perspective supports that proposed by Duval. Duval maintained that the process of mathematical thinking required not only the use of representation systems but also cognitive integration of representation systems [10].

Based on Duval's analysis, learning and comprehending mathematics require relatively similar semiotic representations. The construction of mathematical concepts is a process that primarily transforms or activates symbols into *do-able* procedures and considers symbols as *think-able* objects. This procedure involves transferring *actions* of known objects and treating these actions as mental objects that can be manipulated. This cycle of mental construction has been described as *action*, *process* and *object* [11]. That is, people acquire the existence of new mathematical concepts through a series of actions before the concepts are abstracted as an object or static structure. Subsequently, people treat mathematical concepts as an entity when processing the concepts and no longer consider the particulars. The author of this study situated the investigation of process-to-object theories within the context of integrals. Specifically, engineering students' ability to use the relationship between representations to solve integral problems was examined. The observations suggest a reasonable strategy for operationalising the constructs of process- and object-based thinking within the restricted domain of definite integrals. Based on representations of mathematical objects and process-objects, the versatile thinking structure of the concept of definite integrals is shown in Table 1.

Table 1: The versatile thinking structure of the definite integral concept.

Levels	Representations		
	Symbolic	Graphical	Numerical
Procedure	Can compute integral values using an integral formula	Can only calculate areas using symbolic representations	Cannot use a numerical approximation to calculate area
Process	Understands the relationship between the integrand and upper and lower limits of integration	Comprehends the relationship between the area above the x axis and the integral	Can interpret the limiting process of n rectangular area sums
Object	Can interpret and comprehend that a definite integral is an accumulation function	Can interpret and comprehend the relationship between the area and integral	Can interpret and comprehend the limiting process of a Riemann sum

METHODOLOGY

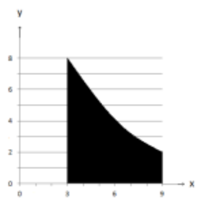
The 25 first-year engineering students who participated in this study were enrolled at a university of technology and had learned the basic rules of integration using primitives, as well as their relationship to the calculation of areas under curves. The instrument used for data collection was a questionnaire containing problems and interviews. The questionnaire comprised seven problems in definite integration. These problems enabled the students' performance regarding the co-ordination of registers and the level of process/object to be analysed. The results of the questionnaire necessitated further investigation into the versatile mathematics thinking of students.

Task 1: If $\int_1^3 f(t)dt = 8.6$, use two strategies to evaluate the value of $\int_2^4 f(t-1)dt$.

Task 2: Is it true or false that if $\int_a^b f(x)dx \geq \int_a^b g(x)dx$, then $f(x) \geq g(x)$ for all $x \in [a,b]$? Justify your answer.

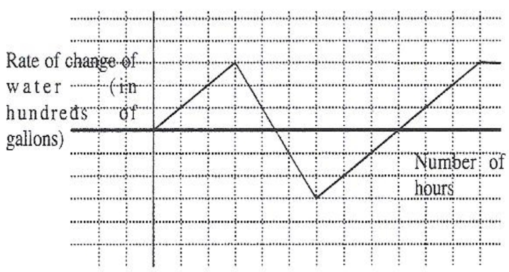
Task 3: If $\int_1^5 f(x)dx = 10$, use two strategies to evaluate the value of $\int_1^5 (f(x)+2)dx$.

Task 4: Use two strategies to calculate $\int_{-3}^3 |x+2|dx$.



Task 5: Estimate the area of the shaded region.

Task 6: Let *f* represent the rate at which the amount of water in Phoenix's water tank changed (hundreds of gallons per hour) over a 12 h period from 6 am to 6 pm last Saturday (Assume that the tank was empty at 6 am (t = 0)). Use the graph of f provided below to answer the following questions: a) How much water was in the tank at noon?; b) What is the meaning of $g(x) = \int_0^x f(t)dt$?; and c) What is the value of $g(9)$?



All questionnaires and problem-solving drafts were independently analysed by the study author and a mathematics scholar after the interviews. Results of the analysis were reviewed and discussed by the author and the scholar until a consensus was reached. To assess and interpret the versatile thinking of the definite integral concept of the engineering students in this study, the versatile thinking structure for definite integrals was constructed. This structure was employed to develop clear standards, and using the triad mechanism proposed by Piaget and Garcia to describe and, then, classify the thinking of the students into various levels [12].

The standards were related to the thinking process adopted and presented by the students when solving problems, as well as their potential to construct relationships among the various representations and properties, and the degree to which they integrated these relationships into their explanation of problem-solving. This mechanism divides the versatile thinking of concepts into three stages. At the intra-operational or intra stage, one tends to focus on a single cognitive item, overlooking other actions, processes and objects of a similar nature.

For example, students believed that they had to perform integral operations to determine $\int_a^b f(x)\,dx$. At the inter-operational or inter stage, one can perceive and confirm the relationships among various actions, processes and objects; however, these items may still appear independent of each other. For example, students understood the relationship between the integral and area, but were unable to differentiate between the relationships of areas above and below the x axis with the integral. At the trans-operational or trans stage, one can use the relationships identified in the previous stage to construct a consistent structure based on the relationships among various actions, processes and objects [13].

The importance of this consistency lies in the fact that it determines the scope of mathematical thinking. According to these data analyses, the conditions of the students' versatile thinking were recorded using nine boxes. For example, if a student only understood *symbol-procedure*, a dot was marked in this box. If a student understood both *symbol-procedure* and *symbol-process*, a line segment was drawn to connect the two boxes. This method was adopted to verify the distribution of the students' versatile thinking structure regarding the definite integral concept. Results of the analysis indicated that the versatile thinking distribution of 25 students could be categorised to three groups.

RESULT AND DISCUSSION

Because students' versatile thinking of the definite integral could be reasonably understood regarding the triad stages, the responses to the task interviews were evaluated, searching for evidence of the intra-, inter-, and trans-operational stages.

Versatile Thinking of the Concept of Integrals in the Intra Stage

The initial level of versatile thinking, when co-ordinating the construction of developmental chains of process/object and representation is the intra stage. Nine students were categorised into this group. One of the versatile thinking characteristics shared by these students (Table 2) was that they could not recognise the relationship between the area and the integral.

Table 2: Characteristics of versatile thinking in the intra stage.

Task	Characteristics of versatile mathematics thinking
1	The students could only think using symbolic representations and did not use graphical representations. Additionally, they may have employed incorrect symbols; for example, $\int_1^3 f(t)\,dt = f(3) - f(1)$. Their concept level of symbolic representation had not reached the process stage level.
2	The students thought with symbolic representations and did not use graphical representations. They assessed the authenticity of the problem by listing examples or manipulating symbols. Their concept level for symbolic representation had not reached the process stage level.
3	The students could only think using symbolic representations and did not employ graphical representations. Their concept level for symbolic representation had not reached the process stage level.
4	The students could only compute integral values using symbolic representations and did not employ graphical representations. Their graphical representation concept level had not reached the process stage level.
5	The students did not use numerical approximations to compute area and attempted to locate curve equations; for example, let $f(x) = ax^2 + bx + c$. Their graphical representation concept level had not reached the process stage level.
6	The students did not answer this problem or attempted to develop a graphical equation; for example, let $f(x) = ax + b$. Their numerical representation and graphical representation concept levels had not reached the process stage level.

The students could solve a number of problems by simply applying rules that had been memorised and in some cases, incorrectly remembered. Consider the following excerpt from the interview conducted by the researcher (R) with Porter (P), who has a collection of rules that enable him to integrate fundamental functions. However, Porter could not solve the problems using graphical representations. In Task 2, he stated that the proposition was true and provided specific examples of functions as evidence without giving graphic representations, failing to provide suitable justifications. He provided a specific example that defined two functions $f(x) = x^2+1$ and $g(x) = x^2$, and calculated two integrals between 1 and 2 to obtain 10/3 for f and 7/3 for g. Subsequently, the interview progressed as shown below:

R: Can you provide a geometric or numerical example?
P: (Draws the two curves of $f(x) = x^2+1$ and $g(x) = x^2$) Like this?
R: Can you do this in graph form without an equation?
P: How do I draw graphs without equations?

Drawing graphs based on the two example functions provided, Porter was unable to think using graphical representations without algebraic formulae. Therefore, Porter's thinking pattern relied on symbolic but not graphical representation. A similar situation occurred for Task 5. Porter did not use a numerical approximation to compute the area; instead, he assumed that the graphical function was a parabola (Figure 2). Subsequently, the interview progressed as shown below:

R: What do you think of this problem?
P: We have to calculate the enclosed area for the parabola, x = 3, x = 9, and the x axis in this problem.
R: Why is it a parabola?
P: Because the graph looks like a parabola.
R: Is it absolutely necessary to compute the area using the integral?
P: Of course. The integral is used to calculate area.

To Porter, an integral was simply a tool for computing area. Another student, John, was also categorised into the intra stage.

For Task 6, he calculated the linear equation that passed through the two points (-2, -2) and (1/2, 8) as 4 x – y + 6 = 0. Then, he calculated the area enclosed by linear lines $\int_{-2}^{1/2}(4x+6)dx + \int_{1/2}^{3} \alpha x dx$, and did not solve the problem using the relationship between the area and integral.

In Task 4, John defined the step function and, then, established two integrals (between [–3, 0] and between [0, 3]). When he was asked: *Why do you separate the area into these two integrals?* He answered: *I separate the area into two integrals because of the absolute values; one integral represents the value to the left of 0, and the other represents the one to the right of 0*. The students in the intra stage generally used a single representation, and symbolic representation was used to solve all types of problems. This indicates that students consider symbolic representation as a support tool. The high proportion of symbolic representations used in versatile thinking has attracted attention. Additionally, students in this group were inclined to rely on analytical thinking instead of visual thinking. They were incapable of visualising problems. Furthermore, they tend to be cognitively fixed on algorithms and procedures instead of recognising the advantages of visualising the tasks.

Versatile Thinking of the Concept of Integrals in the Inter Stage

The next level of versatile thinking of the concept of definite integral regarding the existence of cognitive links and awareness of these links is the inter stage. Fourteen students were categorised into this group. These students understood the relationships between representation systems and could change or transfer the representations in some of the representation systems. However, these students had difficulty co-ordinating these relationships. The students in this group shared the following characteristics (Table 3).

Table 3: Characteristics of versatile thinking in the inter stage.

Task	Characteristics of versatile mathematics thinking
1	The students could think using correct symbolic representations and were capable of taking advantage of the relationship between the area and integral. Their concept level for symbolic representation had reached the process concept level.
2	The students could think using graphical representations, but did not understand the relationship between the area and integral values. They had not reached the process concept level.
3	The students could think using symbolic representations and also solve problems by combining the mean value theorem for integrals or graphical representations. Their concept level for symbolic representation had reached the process concept level.
4	The students could correctly think using symbolic representations and solve problems using graphical representations. They also understood the relationship between the area above the x axis and integral value. However, their functions and graphs may have been incorrect.
5	The students used numerical approximations or upper and lower limits of integration to compute areas. They had reached the process concept level.
6	The students used numerical approximations to compute areas, and could convert graphical representations into numerical representations. However, they had not reached the process concept level.

Helen was one of the students in this group. She could use correct symbolic representations to perform mathematical thinking and could manipulate the area using graphical representations according to the changes in integral symbols in Tasks 1 and 3. However, for Task 2, she says that the proposition is false and gave graphic representations, but failed to make suitable justifications:

R: Can you explain what you think of this task?
H: The area enclosed by f, x = a, x = b, and the x axis is greater than the area enclosed by g, but the function value of f is smaller than g.
R: But the question involves the integral of f being greater than that of g.
H: The integral value is the area; therefore, a greater integral means a greater area.
R: Does this have any relevance to the area being above or below the x axis?
H: It is irrelevant to the area being above or below the x axis.

Similar to Helen, although the students in this group could convert symbolic representations and graphical representations, they believed that an integral value was the same as area. Although they understood the relationship between the area above the x axis and the integral, they did not understand the relationship between the area below the x axis and the integral. In Task 5, Javi divided the region into sections based on the upper and lower limits and, then, calculated the area (Figure 6). He co-ordinated the given graphic representation with the algebraic representation he established. That the function did not have an algebraic expression did not prevent him solving the problem. However, his selections of the height of the rectangle were limited to the left and right endpoints and the midpoint.

R: Tell me what you think of this problem?
J: No curve equation is provided for this problem; therefore, I segmented the area into six rectangles. The sum of the areas of the six rectangles is an approximate value of the original area.
R: Why did you segment the area into six rectangles?
J: Because I had insufficient time. If I had, I would have segmented the area into 10, 12 or even more rectangles.
R: What difference does it make to segment the area into six or 12 rectangles?
J: The more rectangles I have, the more accurate my answer will be.
R: Do you have other methods that can be used to determine the height of the rectangle aside from the function values of the left and right endpoints?
J: The function value of the midpoint can also be used as the height of the rectangle.

The interview results show that students in this group have begun to co-ordinate the representation systems of definite integrals. These students can perform representation treatments in separate representation systems and generalise, abstract or interiorise these procedures into processes. Additionally, they can also perform representation conversions in some of the representation systems and interpret the significance of definite integrals in various representation systems. The challenges for these students are that their understanding of the relationship between integral and area has not yet reached the object concept level, and their versatile thinking of the integral concept remains unstable.

Specifically, they have not yet established stable connections for representation conversions between symbolic representation systems and graphical representation systems. Students in this group differ from those in the intra stage in that they have developed visual methods to better *see* mathematical concepts and problems; although their visual thinking inclines toward local not global thinking and this restricted visualisation actually hinders their solving of the tasks. Additionally, their chosen visualisation only reflects one aspect of the integral concept, which has a number of significant consequences on their ability to solve the other tasks. Nonetheless, visualisation can be used as a tool for connecting or converting between representations.

Versatile Thinking of the Concept of Integrals in the Trans Stage

Two students were categorised into this group. These students could recognise the relationships among representation systems and convert representations between representation systems. The shared characteristics of these students were that their understanding of numerical, graphical and symbolic representation systems approached the object concept level, and that they could perform treatments and conversions on the three representations at the procedure and process concept levels. Kevin used correct symbolic representations to perform mathematical thinking. He also manipulated the area using graphical representations according to the changes in the integral symbols. Consider Task 3 for example, Kevin actually employed three methods to solve the problem.

For Task 2, Kevin stated that the proposition was false and provided graphical representations. Unlike students in the intra stage, who could only apply symbolic representation thinking, Kevin could employ graphical representation as a thinking tool. Additionally, Kevin clearly understood that the area above the x axis was the integral value, and he understood the relationship between the area under the x axis and the integral. As demonstrated by his answer to Task 5, Kevin understood that by segmenting the area into additional rectangles, the sum of the area of the rectangles was closer to the area of the region.

Additionally, he understood that the selection of the height of the rectangle had an infinite number of possibilities. He said, *selecting the left and right endpoints and the midpoint can facilitate calculation; however, the function value of any point can be used as the height of the rectangle*.

As demonstrated by his answer to Task 6, Kevin clearly understood the conversions of numerical, symbolic and graphical representations for the definite integral.

R: How did you calculate the amount of water at noon?
K: I calculated the area from six o'clock in the morning to noon.
R: Why did you calculate the area?
K: Because the y axis represents the variability of the water inflow and the x axis represents the time elapsed. The product of the two is the volume of the water inflow, which is the area.
R: Why did you subtract 225 from 675?
K: Because 675 is the area above the x axis, which represents the amount of water that flows into the tank. Whereas 225 is the area below the x axis, which represents the amount of water that flows out of the tank. Thus, 675 minus 225 equals the amount of water in the tank at noon.

Kevin clearly described the process concept of definite integrals and, after encapsulating the process as objects, he also performed seamless representation treatments and conversions for, or based on, the objects. The most significant difference between the two students in this group and students in the other groups was that the two students in this group had the ability to perform representation treatments in representation systems, and they could perform representation conversions among various representation systems. This ability may involve or be related to visualisation. For Kevin, visualisation is a powerful tool to explore mathematical problems and to ascribe meaning to the concept of definite integrals and the relationship between them.

CONCLUSIONS

In this study, first-year engineering students at universities were recruited as research participants to investigate versatile thinking in the concept of definite integrals, using versatile thinking as a theoretical structure. At the intra stage, students memorise or remember a number of integral methods but do not understand the relationship between the area and the definite integral. They can perform representation treatments for individual representation systems, but cannot generalise, abstract, or interiorise these procedures into processes. Consequently, they lack the ability to convert representations among representation systems.

Additionally, the difficulty these students encounter is the procedural thinking of symbolic representation, not the visual thinking that combines graphical representation. For example, students at the intra stage require formulae and equations to calculate areas and perform mathematical thinking. They regard the definite integral simply as a tool for calculating areas enclosed by curves; thus, the versatile thinking of these students remains at the procedural thinking level. The majority of these students were at the inter stage. They had learned the rules for computing definite integrals and had begun recognising various interrelationships.

Furthermore, they could convert representations in some of the representation systems. The challenge these students faced was that they tended toward process thinking and not object thinking and their visual thinking patterns inclined toward local thinking and were restricted to specific aspects of the integral concept. Only two students were at the trans stage. These students had learned the rules of calculating derivatives and could recognise interrelationships. They also had visualisation abilities and could co-ordinate various representations of mathematical schemas, the concept of limit and definite integrals.

Results of the data analysis show that the main obstacles preventing students from freely shifting within the structure of versatile thinking for the concept of definite integrals were that they had not achieved the process concept for graphical representations of definite integrals, and that this ability involves visualising the abstracted relationships and non-figural information into visual representations and imagery. Based on the students' performance, it can be concluded that visual thinking plays a key role in the development of students' versatile thinking. Visualisation is important for versatile thinking, because it promotes versatile thinking and encourages students to consider problems holistically before dividing them into parts.

From a didactic perspective, the introduction of graphics that illustrate a specific case and a counter-example may focus attention on the key aspects of the representational relationship and render the graphics an integral part of the concept of definite integrals. However, adopting a specific method of teaching versatile thinking raises the following questions: What conversion processes are involved in moving versatility among various mathematical representations, including those of a visual nature? How can visual, mixed and non-visual methods be combined in class to improve the versatile thinking of students?

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