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Modelling structures as systems of springs

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ABSTRACT: The basic physics of a spring and elementary structural analysis principles are combined to produce a powerful procedure to analyse static structures. Structural members are modelled as springs that are assembled in parallel and/or series configurations to represent a structure's equivalent stiffness. The overall deflection of the structure can then be computed and, after an appropriate deconstruction of the spring system, internal member forces can be determined. This procedure is effectively presented in some structural dynamics texts, but it is not taught when analysing or designing static structures. The authors feel that this procedure should be taught early in a civil engineer's undergraduate education to facilitate the understanding of structural behaviour in a wider variety of structures courses. The procedure also provides an effective means to check traditional structural analysis calculations, assists in the design of structural members and/or structural systems, and can be used to verify finite element results.

INTRODUCTION

Basic physics and elementary structural analysis principles are used to develop a straightforward but powerful procedure to analyse static structures. The methodology is based on modelling structural members as springs and assembling these springs in a configuration that accurately represents the structural system; the overall effective stiffness of the structure can then be computed so that the overall drift (deflection) a structure experiences when subjected to applied static loads can be determined. This procedure, or some variant of it, is effectively presented in some structural dynamics texts (ie [1]), where computing the effective stiffness of a structure is required when modelling its dynamic response with a single or a few degrees of freedom. Further, the spring system can be methodically deconstructed to determine the internal forces in individual springs, which represent the internal forces in individual structural members.

Unfortunately, this procedure is rarely used in elementary statics-based structural analysis or design courses or presented in textbooks (ie [2] and [3]) developed for these classes. As a result, students are less able to link fundamental mechanics concepts to the behaviour of complex structural systems. Moreover, they lack an effective means of checking structural calculations obtained from more traditional techniques.

A reasonably complex building structure will be quickly and effectively analysed in this paper after reviewing the simple physics of a spring, developing an understanding of springs in parallel and series, presenting stiffness representations of common structural members, and exercising engineering judgement in constructing spring models. While the results are approximate due to connection and rigid member assumptions, they can be used to check traditional structural analysis calculations, assist in the design of structural members and/or structural systems, and verify finite element results.

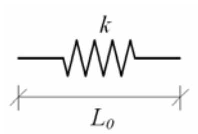
BASIC PHYSICS OF A SPRING

Hooke's Law

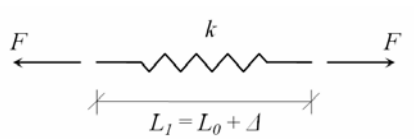
The relation that describes how a linear elastic spring stretches or compresses due to a force is presented in basic physics texts and is known as Hooke's Law; mathematically, it is given by:

$$F = k\Delta \qquad (1)$$

This is an equation of a straight line that passes through the origin of its load deflection plot, where F = a tensile or compressive force applied to the spring, k = the spring's stiffness (or spring constant), and Δ = the deflection of the spring. F and Δ are both positive if the spring is subjected to a tensile force and both negative if it is subjected to a compressive force. Figure 1a shows an unloaded spring with stiffness k and length L_0, while Figure 1b shows a spring under tension.



(a) Unloaded spring



(b) Spring under tension

Figure 1: Springs before and after loading (1a-1b).

Figure 1b is the same spring subjected to a tensile force F, where the loaded length L_1 is greater than the unloaded length L_0 since Δ is positive when the spring is in tension. If the spring were to be loaded in compression, the loaded length L_1 would be less than the unloaded length L_0.

Relationship to a Simple Structure

One of the simplest structural members that is not literally a spring itself is a hanger (a structure of this type is also often called a bar or strut) (see Figure 2).

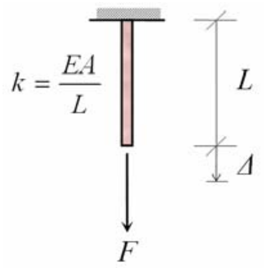


Figure 2: Hanger (bar) under load.

If an axial force F is applied at the end of the hanger, the axial deflection of its tip Δ can be derived using undergraduate mechanics of material concepts, thus:

$$\Delta = \frac{FL}{EA} \tag{2}$$

where E = Young's modulus of the material, A = cross sectional area of the bar and L = the original (unloaded) length of the bar. If, however, this equation is arranged to be of the form of Hooke's equation:

$$F = \frac{EA}{L}\Delta \tag{3}$$

the equivalent stiffness of the hanger is observed to be:

$$k_{eq} = \frac{EA}{L} \tag{4}$$

The primary point here is that the hanger can be considered as a spring with equivalent stiffness $k_{eq} = EA/L$, a concept that is rarely taught to students for application in requisite structural analysis and design courses.

CONCEPT OF A SPRING SYSTEM

Obviously, it is of interest to model more complex structures than the simple hanger presented in the previous section. In order to do so, one must be able to represent a general structural system as an assemblage of springs in parallel and in series. Considering the form of Hooke's equation, the equivalent stiffness of the entire system can be determined and subsequently used to compute the deflection of the global system (structure) in response to the applied forces.

Springs in Parallel

When an assemblage of springs is subjected to a force and all springs deflect by the same amount, the springs are said to be in parallel. Figure 3 indicates why the term *parallel* is given to these springs – all springs in the assemblage are arranged parallel to one another.

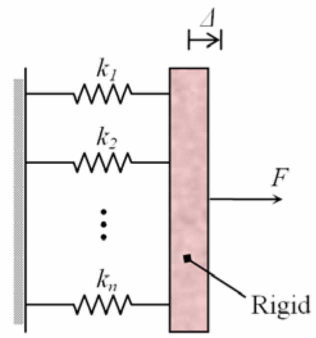


Figure 3: n springs in parallel.

The equivalent stiffness for n springs in parallel is obtained simply by adding together all the individual stiffnesses, thus:

$$k_{eq} = k_1 + k_2 + \ldots + k_n \tag{5}$$

Springs in Series

When an assemblage of springs is subjected to a force and all springs in general deflect by different amounts, the springs are said to be in series. Figure 4 shows n springs in series, where they are connected together in a chainlike fashion or in *series*.

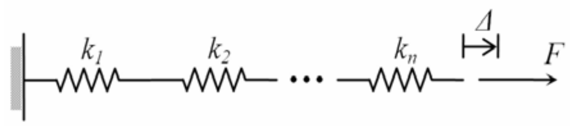


Figure 4: n springs in series.

The equivalent stiffness for n springs in series is:

$$k_{eq} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n}} = \frac{k_1 \cdot k_2 \cdot \cdots \cdot k_n}{k_1 + k_2 + \cdots + k_n} \tag{6}$$

General Spring System

All linear elastic structures can be modelled as a system of springs that are, in general, in parallel and/or in series with each other. For example, in Figure 5, springs k_2 and k_3 are in parallel with each other, and their equivalent stiffness is $k_2 + k_3$; this equivalent stiffness is in series with spring k_1 and the equivalent stiffness of the entire three-spring system is:

$$k_{eq} = \frac{k_1 \cdot (k_2 + k_3)}{k_1 + k_2 + k_3} \tag{7}$$

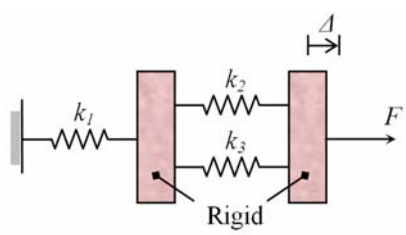


Figure 5: Parallel-series spring system.

The deflection at the location of the applied load relative to the left-end of the structure (a fixed support) can be computed by substituting the system's equivalent stiffness into Hooke's Law and rearranging to solve for *Δ*; that is:

$$\Delta = \frac{F}{k_{eq}} = \frac{F(k_1 + k_2 + k_3)}{k_1 \cdot (k_2 + k_3)} \tag{8}$$

STRUCTURAL MEMBERS AND STRUCTURAL SYSTEMS MODELLED WITH SPRINGS

Additional representative spring equations are needed to model common building structural members. These spring equations can be derived using structural analysis principles; however, the vast majority of students in structural analysis are probably unaware that they have this capability.

Engineering Judgement

It must be emphasised early on that good engineering judgement is required to model effectively a structure while keeping the hand computations tractable. In order to simplify the analysis, some structural members can be assumed to be perfectly rigid (a spring with an infinite stiffness) with little loss of accuracy, while the relative flexibility of other members must be considered and, therefore, modelled as springs. Given the applied loads and the properties of the structural elements (ie material properties, section properties, structural dimensions, etc), the analyst is required to carefully examine the structural drawings (or sketches) and make assumptions about structural connections, know how to model individual structural components as springs, and assemble the springs in parallel and/or series to produce a representative equivalent stiffness so the desired structural deflections and internal member forces can be computed.

Rigid Members

When a structural member is very stiff relative to more flexible structural members in its vicinity, it can often be considered to be perfectly rigid with little resulting error. For example, floor and roof structural systems (ie stiff floor/roof diaphragms) are commonly considered to be rigid relative to the columns that connect one diaphragm to another.

Flexible Members

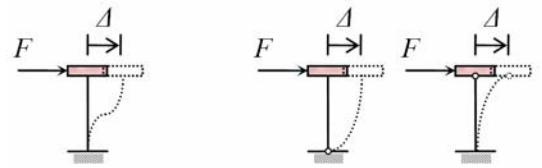
Basic structural analysis principles allow an analyst to model a column as a spring. A column that is rigidly connected to a rigid diaphragm – the ends of the column do not rotate relative to the diaphragm – at its base and top, has an equivalent lateral stiffness (see Figure 6a) given by:

$$k_{eq} = \frac{12EI}{L^3} \tag{9}$$

where I = a section property of the column known as the moment of inertia and L = the length of the column. This equation represents the stiffness due to bending in the column and is used when the top of the column displaces laterally relative to its base.

However, if either the top or bottom of the column is perfectly free to rotate relative to the diaphragm – a pinned-connection – and the other connection is rigid, the stiffness afforded by the column is reduced by a factor of four (see Figure 6b); that is:

$$k_{eq} = \frac{3EI}{L^3} \tag{10}$$



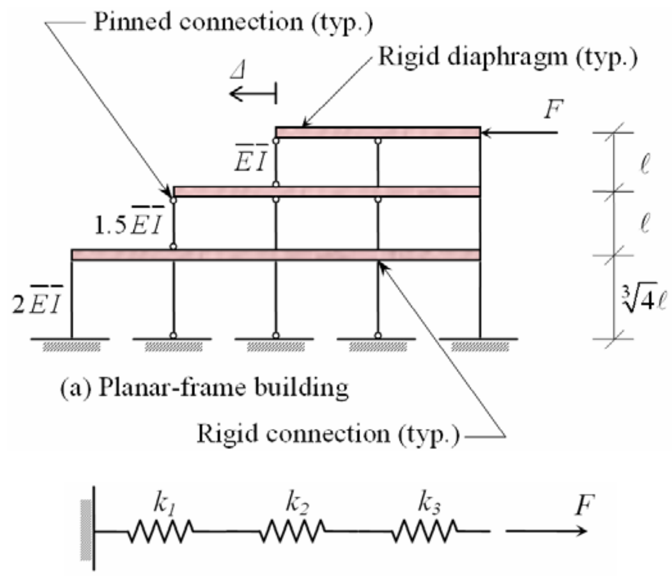
(a) Rigid-rigid (b) Pinned-rigid

Figure 6: Column to diaphragm connections.

In actuality, connections are never perfectly rigid or pinned, but structural deflections can be reasonably approximated using these connection approximations. Finally, if both the top and the bottom of a column is pinned (free to rotate relative to the diaphragm to which it is connected), the stiffness provided by the column in the lateral direction is zero; that is, the boundary conditions of the column in this case render the column ineffective in resisting lateral loads.

Example: Planar Frame Building Structure

Consider the three-storey, planar-frame, building structure in Figure 7a. It is of interest to compute the drift of the building *Δ* (deflection at roof level) given that the applied loading, the structural geometry, and member material and section properties are known. Due to the assumed rigid floor and roof diaphragms, the springs representing each of the columns at a storey level are in parallel since they are all forced to deflect the same amount.



(b) Equivalent spring

Figure 7: Hypothetical 2D building example (7a-7b).

All five columns of the first storey have flexural rigidity $2\overline{EI}$ and equal lengths as shown in Figure 7a; individual stiffnesses are different, however, due to the different end (boundary) conditions. Since all five springs are in parallel, the first-storey stiffness is:

$$k_1 = 2 \times \frac{12(2\overline{EI})}{\left(\sqrt[3]{4}\ell\right)^3} + 3 \times \frac{3(2\overline{EI})}{\left(\sqrt[3]{4}\ell\right)^3} = \frac{48}{4}\frac{\overline{EI}}{\ell^3} + \frac{18}{4}\frac{\overline{EI}}{\ell^3}$$

$$= \frac{66}{4}\frac{\overline{EI}}{\ell^3} \qquad (11)$$

Similarly, the second-storey springs are all in parallel and the storey stiffness is:

$$k_2 = 0 + 2 \times \frac{3(1.5\overline{EI})}{\ell^3} + \frac{12(1.5\overline{EI})}{\ell^3} = \frac{9\overline{EI}}{\ell^3} + \frac{18\overline{EI}}{\ell^3}$$

$$= \frac{27\overline{EI}}{\ell^3} \qquad (12)$$

The left-most column of the second-storey has zero stiffness since both ends of the column are pinned.

The third-storey stiffness is:

$$k_3 = 0 + \frac{3\overline{EI}}{\ell^3} + \frac{12\overline{EI}}{\ell^3} = \frac{15\overline{EI}}{\ell^3} \qquad (13)$$

Since each storey is in series relative to its neighbouring storey, the equivalent stiffness of the entire structure, shown schematically in Figure 7b, is given by:

$$k_{eq} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}} = \frac{1}{\frac{4\ell^3}{66\overline{EI}} + \frac{\ell^3}{27\overline{EI}} + \frac{\ell^3}{15\overline{EI}}}$$

$$= \frac{1}{\frac{4392\ell^3}{26{,}730\overline{EI}}} = \frac{26{,}730\overline{EI}}{4392\ell^3} \qquad (14)$$

Thus, rearranging Hooke's Law, the drift is given by the following equation:

$$\Delta = \frac{F}{k_{eq}} = \frac{4392F\ell^3}{26{,}730\overline{EI}} \qquad (15)$$

Internal forces in any column at each storey can be computed by proportion of the individual column stiffness relative to the overall stiffness at the storey. For example, for the load case shown in Figure 7, the force at any storey level is F. If the *lateral* force in the leftmost column of the first storey is to be computed, it is equal to F times the percentage of the stiffness this column possesses relative to the total storey stiffness; the force in this column is:

$$F_{L1} = \frac{k_{L1}}{k_1}F \qquad (16)$$

where k_{L1} is the stiffness of the leftmost column of the first storey and k_1 is the total first storey stiffness, and:

$$k_{L1} = \frac{12(2\overline{EI})}{\left(\sqrt[3]{4}\ell\right)^3} = \frac{24}{4}\frac{\overline{EI}}{\ell^3} \qquad (17)$$

and:

$$k_1 = \frac{66}{4}\frac{\overline{EI}}{\ell^3} \qquad (18)$$

so:

$$F_{L1} = \frac{24}{66}F = \frac{12}{33}F \qquad (19)$$

or approximately one third of the storey force F.

CONCLUSIONS

Modelling linear elastic structures as a system of springs enhances the understanding of structural behaviour. The procedure also provides a quick and effective means to check traditional structural analysis calculations, assists in the design of structural members and/or structural systems, and can be used to verify finite element results.

This modelling procedure, or some variation of it, is effectively presented in some structural dynamics texts for computing a structure's equivalent stiffness but, unfortunately, it is rarely used in elementary statics-based structural analysis or design courses. The authors feel that this procedure, if taught early in a civil engineer's undergraduate education, will facilitate the understanding of structural behaviour in a wider variety of structures courses.

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